

# Integrable multi-component generalization of a modified short pulse equation

Yoshimasa Matsuno\*

*Division of Applied Mathematical Science,  
Graduate School of Sciences and Technology for Innovation  
Yamaguchi University, Ube, Yamaguchi 755-8611*

We propose a multi-component generalization of the modified short pulse (SP) equation which was derived recently as a reduction of Feng's two-component SP equation. Above all, we address the two-component system in depth. We obtain the Lax pair, an infinite number of conservation laws and multisoliton solutions for the system, demonstrating its integrability. Subsequently, we show that the two-component system exhibits cusp solitons and breathers for which the detailed analysis is performed. Specifically, we explore the interaction process of two cusp solitons and derive the formula for the phase shift. While cusp solitons are singular solutions, smooth breather solutions are shown to exist, provided that the parameters characterizing the solutions satisfy certain condition. Last, we discuss the relation between the proposed system and existing two-component SP equations.

**KEYWORDS:** Modified short pulse equation, multi-component generalization, loop soliton, breather

---

\*E-mail address: matsuno@yamaguchi-u.ac.jp

## I. INTRODUCTION

In a recent development of the theory of solitons, the short pulse (SP) equation has attracted considerable attention. It can be written in an appropriate dimensionless form as

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \quad (1.1)$$

where  $u = u(x, t)$  represents the magnitude of the electric field and subscripts  $x$  and  $t$  appended to  $u$  denote the partial differentiation. The SP equation has been proposed as a model equation describing the propagation of ultrashort optical pulses in nonlinear media.<sup>1</sup> In the context of nonlinear optics, the cubic nonlinear Schrödinger (NLS) equation has played a central role in studying the dynamics of optical solitons. While the NLS equation is applicable to the evolution of the slowly varying envelope, the SP equation works for short waves whose spectra are not localized around the carrier frequency. A numerical analysis shows that as the pulse length shortens, the SP equation becomes a better approximation to the solution of the Maxwell equation when compared with the prediction of the NLS equation.<sup>2</sup> We recall that the SP equation has been derived for the first time in an attempt to constructing integrable partial differential equations (PDEs) associated with pseudospherical surfaces.<sup>3,4</sup> The integrability aspects of the SP equation have been studied from various mathematical points of view.<sup>5-7</sup> Of particular interest is the existence of breather solutions whose characteristics are different from those of envelope soliton solutions of the NLS equation.<sup>8,9</sup> See also a recent article which surveys the exact method of solution for the SP equation and the properties of soliton and periodic solutions.<sup>10</sup>

To take into account the effects of polarization or anisotropy, the SP equation has been generalized to the multi-component integrable systems. Among them, Matsuno proposed the two-component system<sup>11</sup>

$$u_{xt} = u + \frac{1}{2}(uvu_x)_x, \quad v_{xt} = v + \frac{1}{2}(uvv_x)_x, \quad (1.2)$$

which is a special case of the following coupled nonlinear PDEs for the  $n$  variables  $u_i = u_i(x, t)$ , ( $i = 1, 2, \dots, n$ ):

$$u_{i,xt} = u_i + \frac{1}{2}(Fu_{i,x})_x, \quad (i = 1, 2, \dots, n), \quad (1.3a)$$

with

$$F = \frac{1}{2} \sum_{1 \leq j, k \leq n} c_{jk} u_j u_k. \quad (1.3b)$$

Here,  $c_{jk}$  are arbitrary constants with the symmetry  $c_{jk} = c_{kj}$  ( $j, k = 1, 2, \dots, n$ ). Obviously, the identification  $u_1 = u$  and  $u_2 = v$  with  $c_{11} = c_{22} = 0$ ,  $c_{12} = 1$  in (1.3) yields (1.2).

Another integrable generalization is due to Feng, which is given by<sup>12</sup>

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx} + \frac{1}{2}v^2u_{xx}, \quad v_{xt} = v + \frac{1}{6}(v^3)_{xx} + \frac{1}{2}u^2v_{xx}. \quad (1.4)$$

If we identify  $v$  with  $u$ , then the system (1.2) degenerates to the SP equation (1.1) while Feng's system (1.3) recasts to (1.1) upon putting  $v = 0$ .

Quite recently, Sakovich considered the integrability of the nonlinear PDE<sup>13</sup>

$$u_{xt} = u + au^2u_{xx} + buu_x^2, \quad (1.5)$$

where  $a$  and  $b$  are arbitrary constants. Sakovich observed that the case  $a/b = 1/2$  corresponds to the SP equation (1.1) whereas the case  $a/b = 1$  yields, after rescaling the variable  $u$ , a new nonlinear PDE

$$u_{xt} = u + \frac{1}{2}u(u^2)_{xx}. \quad (1.6)$$

Hereafter, we call Eq. (1.6) the modified SP equation. Note that Eq. (1.6) follows simply from Feng's system (1.4) by putting  $v = \pm u$  and hence its integrability will be assured.

The main objective of the present paper is to generalize Eq. (1.6) to an integrable multi-component system. Specifically, we propose the following coupled PDEs for the  $n$  variables  $u_i = u_i(x, t)$ :

$$u_{i,xt} = u_i + (Fu_{i,x})_x - \frac{1}{2} \left( \sum_{1 \leq j, k \leq n} c_{jk} u_{j,x} u_{k,x} \right) u_i, \quad (i = 1, 2, \dots, n), \quad (1.7)$$

where  $F$  is given by (1.3b). For the special case of  $n = 2$ , we put  $u_1 = u$  and  $u_2 = v$  and  $c_{11} = c_{22} = 0, c_{12} = 1$  and see that the system (1.7) reduces to the two-component system

$$u_{xt} = u + v(uu_x)_x, \quad v_{xt} = v + u(vv_x)_x. \quad (1.8)$$

Eq. (1.6) is obtained if one puts  $v = u$  in (1.8) and hence one can regard the system (1.7) as a multi-component generalization of the modified SP equation.

The present paper is organized as follows. In Sec. II, we develop an exact method for solving the modified SP equation. Specifically, we employ a direct method (or the bilinear transformation method) combined with a hodograph transformation which worked well for the analysis of the SP equation.<sup>9</sup> We present the parametric representation of the multisoliton solutions, and show that they are closely related to the multisoliton solutions of the sine-Gordon equation. The properties of solutions are briefly described. We also demonstrate that the modified SP equation transforms to the SP equation through a hodograph transformation. In Sec. III, we generalize the modified SP equation to the multi-component system (1.7) and provide its multisoliton solutions. The basic strategy is to start from a multi-component version of the bilinear form associated with the modified SP equation and then transform it back to the system of equations for the original variables through appropriate dependent variable transformations. In Sec. IV, we analyze the two-component system (1.8) in detail. We present its Lax pair, an infinite number of conservation laws and multisoliton solutions. We show that the system supports cusp solitons and breathers, and investigate their properties. Specifically, we address the interaction process of two cusp solitons, proving its solitonic feature. Subsequently, we deal

with the one-breather solution for which a condition for the existence of smooth breather is derived. Section V is devoted to concluding remarks.

## II. MODIFIED SHORT PULSE EQUATION

In this section, we develop a systematic method for solving the modified SP equation (1.6). While an exact method has already been provided for Eq. (1.6),<sup>13</sup> we present an alternative approach which is applicable to the multi-component system. We also show that the SP equation is related to the modified SP equation through a hodograph transformation.

### A. Hodograph transformation

We introduce the hodograph transformation  $(x, t) \rightarrow (y, \tau)$  by

$$dy = rdx + ru^2 dt, \quad d\tau = dt, \quad (2.1a)$$

where  $r(> 0)$  is a function of  $u$  to be determined later. Then, the  $x$  and  $t$  derivatives transform as

$$\frac{\partial}{\partial x} = r \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + ru^2 \frac{\partial}{\partial y}. \quad (2.1b)$$

It turns out that the variable  $x = x(y, \tau)$  obeys a system of linear PDEs

$$x_y = \frac{1}{r}, \quad x_\tau = -u^2. \quad (2.2)$$

The solvability condition of this system, i.e.,  $x_{y\tau} = x_{\tau y}$  yields the evolution equation for  $r$

$$r_\tau = 2r^2 u u_y. \quad (2.3)$$

Applying the transformation (2.1) to the modified SP equation (1.6), one has

$$ru_{y\tau} + r_\tau u_y = u + r^2 u u_y^2. \quad (2.4)$$

If we multiply (2.4) by  $u_y$  and eliminate the variable  $u$  with the help of (2.3), we can put (2.4) into the form of a linear ordinary differential equation for  $u_y^2$

$$\frac{1}{2}(u_y^2)_\tau + \frac{r_\tau}{2r} u_y^2 = \frac{r_\tau}{2r^3}. \quad (2.5)$$

We impose the boundary conditions  $u(\pm\infty, \tau) = 0, r(\pm\infty, \tau) = 1$ . Then, Eq. (2.5) can be integrated with respect to  $\tau$  to give the solution

$$u_y^2 = \frac{1}{r} - \frac{1}{r^2}. \quad (2.6)$$

Using the relation  $u_y = u_x/r$  which follows from (2.1b), we can determine the form of  $r$  in terms of  $u_x$

$$r = 1 + u_x^2. \quad (2.7)$$

It follows from (2.3), (2.4) and (2.6) that

$$u_{y\tau} = \left(\frac{2}{r} - 1\right)u = (2x_y - 1)u. \quad (2.8)$$

The equation (2.8) coupled with the system of equations (2.2) is the starting point in constructing multisoliton solutions.

## B. Parametric representation of multisoliton solutions

We construct the multisoliton solutions of the modified SP equation by means of the direct method.<sup>14,15</sup> To this end, we introduce the following dependent variable transformation for  $u$  and the hodograph transformation for  $x$

$$u = \frac{g}{f}, \quad (2.9)$$

$$x = y + \frac{h}{f}, \quad (2.10)$$

where  $f, g$  and  $h$  are tau-functions which are the basic constituents of soliton solutions. The second equation in (2.2) is then transformed to the bilinear equation

$$D_\tau h \cdot f = -g^2, \quad (2.11)$$

whereas Eq. (2.8) reduces to

$$\frac{2gf_y}{f^3}(f_\tau + h) - \frac{1}{f^2}(f_\tau g_y + f_y g_\tau + f_{y\tau}g + gh_y) + \frac{1}{f}(g_{y\tau} - g) = 0. \quad (2.12)$$

Here, the bilinear operators  $D_\tau$  and  $D_y$  are defined by

$$D_\tau^m D_y^n f \cdot g = \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n f(\tau, y)g(\tau', y') \Big|_{\tau'=\tau, y'=y}, \quad (m, n = 0, 1, 2, \dots). \quad (2.13)$$

We can decouple Eq. (2.12) into a set of equations

$$f_\tau + h = 0, \quad (2.14)$$

$$f_\tau g_y + f_y g_\tau + f_{y\tau}g + gh_y - f(g_{y\tau} - g) = 0. \quad (2.15)$$

Introducing  $h$  from (2.14) into (2.15) and (2.11), we obtain the system of bilinear equations for  $f$  and  $g$ :

$$D_y D_\tau f \cdot g = fg, \quad (2.16)$$

$$D_\tau^2 f \cdot f = 2g^2. \quad (2.17)$$

Then, the expression (2.10) with  $h$  from (2.14) becomes

$$x = y - (\ln f)_\tau, \quad (2.18)$$

which, coupled with (2.9), gives a parametric representation of solutions. The standard procedure in the context of the direct method<sup>14,15</sup> can be applied to obtain soliton solutions of the system of bilinear equations (2.16) and (2.17). However, this problem will be discussed in Sec. IV. Below, we present an alternative approach.

### C. Parametric solutions in terms of tau-functions of the sine-Gordon equation

It has been shown that the soliton solutions of the SP equation are expressed in terms of the tau-functions for the soliton solutions of the sine-Gordon equation. We summarize the method developed in Ref. 9 keeping its application to the modified SP equation in mind. First, solving (2.6) in  $r$ , we find that  $r$  has two roots

$$r = \frac{1}{2u_y^2} \left( 1 \pm \sqrt{1 - 4u_y^2} \right). \quad (2.19)$$

If we put

$$u_y = \frac{1}{2} \sin \phi, \quad \phi = \phi(y, \tau), \quad (2.20)$$

then (2.19) gives

$$r = \frac{1}{\cos^2 \frac{\phi}{2}}, \quad \frac{1}{\sin^2 \frac{\phi}{2}}. \quad (2.21)$$

We choose the former solution, i.e.,  $r = 1/\cos^2 \frac{\phi}{2}$  and introduce this expression and (2.20) into (2.3) to express  $u$  in terms of  $\phi$

$$u = \frac{1}{2} \phi_\tau. \quad (2.22)$$

Substituting (2.22) into (2.20), we see that  $\phi$  satisfies the sine-Gordon equation

$$\phi_{y\tau} = \sin \phi. \quad (2.23)$$

The  $N$ -soliton solution of Eq. (2.23) is expressed in terms of the tau-functions  $f_{sG}$  and  $\bar{f}_{sG}$  as

$$\phi = 2i \ln \frac{\bar{f}_{sG}}{f_{sG}}, \quad (2.24)$$

with

$$f_{sG} = \sum_{\mu=0,1} \exp \left[ \sum_{j=1}^N \mu_j \left( \xi_j + \frac{\pi}{2} i \right) + \sum_{1 \leq j < k \leq N} \mu_j \mu_k \gamma_{jk} \right], \quad (2.25a)$$

$$\bar{f}_{sG} = \sum_{\mu=0,1} \exp \left[ \sum_{j=1}^N \mu_j \left( \xi_j - \frac{\pi}{2} i \right) + \sum_{1 \leq j < k \leq N} \mu_j \mu_k \gamma_{jk} \right], \quad (2.25b)$$

where

$$\xi_j = p_j y + \frac{1}{p_j} \tau + \xi_{j0}, \quad (j = 1, 2, \dots, N), \quad (2.26a)$$

$$e^{\gamma_{jk}} = \left( \frac{p_j - p_k}{p_j + p_k} \right)^2, \quad (j, k = 1, 2, \dots, N; j \neq k). \quad (2.26b)$$

Here,  $p_j$  and  $\xi_{j0}$  are arbitrary complex-valued parameters satisfying the conditions  $p_j \neq \pm p_k$  for  $j \neq k$  and  $N$  is an arbitrary positive integer. The notation  $\sum_{\mu=0,1}$  implies the summation over all possible combinations of  $\mu_1 = 0, 1, \mu_2 = 0, 1, \dots, \mu_N = 0, 1$ . The tau-functions  $f_{sG}$  and  $\bar{f}_{sG}$  from (2.25) solve the bilinear equations

$$D_y D_\tau f_{sG} \cdot f_{sG} = \frac{1}{2}(f_{sG}^2 - \bar{f}_{sG}^2), \quad D_y D_\tau \bar{f}_{sG} \cdot \bar{f}_{sG} = \frac{1}{2}(\bar{f}_{sG}^2 - f_{sG}^2). \quad (2.27)$$

It follows from (2.24) and (2.27) that

$$\cos \phi = 1 - 2(\ln \bar{f}_{sG} f_{sG})_{y\tau}. \quad (2.28)$$

Inserting this expression into the relation

$$x_y = 1/r = \cos^2 \frac{\phi}{2} = \frac{1}{2}(1 + \cos \phi), \quad (2.29)$$

and then integrating (2.29) with respect to  $y$ , we obtain

$$x = y - (\ln \bar{f}_{sG} f_{sG})_\tau + d, \quad (2.30)$$

where  $d$  is an integration constant depending generally on  $\tau$ . If we substitute (2.22) with (2.24) and (2.30) into the second equation in (2.2), then we find

$$D_\tau^2 \bar{f}_{sG} \cdot f_{sG} = d'(\tau) \bar{f}_{sG} f_{sG}. \quad (2.31)$$

We recall, however, that the tau-functions  $f_{sG}$  and  $\bar{f}_{sG}$  satisfy the bilinear equation

$$D_\tau^2 \bar{f}_{sG} \cdot f_{sG} = 0, \quad (2.32)$$

and hence  $d'(\tau) = 0$ , showing that  $d$  is independent of  $\tau$ . Due to the translational invariance of the modified SP equation, we can set this constant zero without loss of generality. It follows from (2.22) and (2.24) that

$$u = i \left( \ln \frac{\bar{f}_{sG}}{f_{sG}} \right)_\tau. \quad (2.33)$$

Consequently, (2.33) and (2.30) with the tau-functions (2.25) give a parametric representation for the  $N$ -soliton solution of the modified SP equation.

If we compare (2.9) and (2.18) with (2.33) and (2.30), respectively, we can infer that

$$f = \bar{f}_{sG} f_{sG}, \quad g = i D_\tau \bar{f}_{sG} \cdot f_{sG}. \quad (2.34)$$

We can verify these relations by showing that (2.34) indeed satisfy the bilinear equations (2.16) and (2.17). Actually, inserting (2.27) into the identity

$$2D_y D_\tau [ab \cdot (D_\tau a \cdot b)] = D_\tau [a^2 \cdot (D_y D_\tau b \cdot b)] + D_\tau [(D_y D_\tau a \cdot a) \cdot b^2], \quad (2.35)$$

with  $a = \bar{f}_{sG}$  and  $b = f_{sG}$ , we obtain

$$2D_y D_\tau [\bar{f}_{sG} f_{sG} \cdot (D_\tau \bar{f}_{sG} \cdot f_{sG})] = D_\tau \bar{f}_{sG}^2 \cdot f_{sG}^2 = 2\bar{f}_{sG} f_{sG} D_\tau \bar{f}_{sG} \cdot f_{sG}, \quad (2.36)$$

which is just (2.16) with  $f$  and  $g$  from (2.34). To proceed, we use the identity

$$D_\tau^2 ab \cdot ab = 2ab D_\tau^2 a \cdot b - 2(D_\tau a \cdot b)^2, \quad (2.37)$$

with  $a = \bar{f}_{sG}$  and  $b = f_{sG}$ . This gives

$$D_\tau^2 \bar{f}_{sG} f_{sG} \cdot \bar{f}_{sG} f_{sG} = 2\bar{f}_{sG} f_{sG} D_\tau^2 \bar{f}_{sG} \cdot f_{sG} - 2(D_\tau \bar{f}_{sG} \cdot f_{sG})^2. \quad (2.38)$$

In view of (2.32), however, the first term on the right-hand side of (2.38) vanishes, giving rise to (2.17). We remark that an alternative proof of (2.34) has been presented by employing a different bilinearization of the sine-Gordon equation from (2.27), i.e.,  $D_\tau(D_\tau D_y - 1)\bar{f}_{sG} \cdot f_{sG} = 0$ ,  $D_\tau^2 \bar{f}_{sG} \cdot f_{sG} = 0$ .<sup>16</sup>

## D. Transformation of the modified short pulse equation

Here, we show that the modified SP equation can be transformed to the SP equation via a hodograph transformation combined with a linear dependent variable transformation. To this end, we introduce the transformations  $u \rightarrow \bar{u}$  and  $(x, t) \rightarrow (\bar{x}, \bar{t})$

$$\bar{u} = 2u, \quad \bar{x} = x - \int_{-\infty}^x u_x^2 dx, \quad \bar{t} = t. \quad (2.39)$$

Using the local conservation law  $(u_x^2)_t = [u^2(1 + u_x^2)]_x$  of the modified SP equation, we rewrite the  $\bar{x}$  and  $\bar{t}$  derivatives in terms of  $x$  and  $t$  derivatives as

$$\frac{\partial}{\partial \bar{x}} = \frac{1}{1 - u_x^2} \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial \bar{t}} = \frac{\partial}{\partial t} + \frac{u^2(1 + u_x^2)}{1 - u_x^2} \frac{\partial}{\partial x}. \quad (2.40)$$

It is straightforward by applying (2.40) to the SP equation to show that

$$\bar{u}_{\bar{x}\bar{t}} - \bar{u} - \frac{1}{2}(\bar{u}^2 \bar{u}_{\bar{x}})_{\bar{x}} = \frac{2}{1 - u_x^2} \left\{ u_{xt} - u - \frac{1}{2}u(u^2)_{xx} \right\}. \quad (2.41)$$

The relation (2.41) implies that if  $u$  solves the modified SP equation, then  $\bar{u}$  solves the SP equation, i.e.,

$$\bar{u}_{\bar{x}\bar{t}} = \bar{u} + \frac{1}{6}(\bar{u}^3)_{\bar{x}\bar{x}}, \quad (2.42)$$

and vice versa.



We recall that the hodograph transformation  $(\bar{x}, \bar{t}) \rightarrow (\bar{y}, \bar{\tau})$

$$d\bar{y} = \bar{r} d\bar{x} + \frac{1}{2} \bar{r} \bar{u}^2 d\bar{t}, \quad d\bar{\tau} = d\bar{t}, \quad \bar{r} = \sqrt{1 + \bar{u}_{\bar{x}}^2}, \quad (2.43a)$$

or, equivalently in terms of the derivatives

$$\frac{\partial}{\partial \bar{x}} = \bar{r} \frac{\partial}{\partial \bar{y}}, \quad \frac{\partial}{\partial \bar{t}} = \frac{\partial}{\partial \bar{\tau}} + \frac{1}{2} \bar{r} \bar{u}^2 \frac{\partial}{\partial \bar{y}}, \quad (2.43b)$$

has been used for solving the SP equation. It follows from (2.40) and  $\bar{u} = 2u$  that  $\bar{u}_{\bar{x}} = 2(1 - u_x^2)^{-1} u_x$ . Substituting this relation into the definition of  $\bar{r}$  from (2.43a) and referring to (2.7), we obtain an important relation which connects  $r$  with  $\bar{r}$ :

$$\bar{r} = \frac{r}{2 - r}. \quad (2.44)$$

We use (2.1b), (2.40), (2.43b) and (2.44) to derive the relation

$$\bar{r} \frac{\partial}{\partial \bar{y}} = \frac{r}{1 - u_x^2} \frac{\partial}{\partial y} = \frac{1 + u_x^2}{1 - u_x^2} \frac{\partial}{\partial y} = \bar{r} \frac{\partial}{\partial y}.$$

Since  $\bar{r} \neq 0$ , this gives  $\partial/\partial \bar{y} = \partial/\partial y$ . Similarly, one has  $\partial/\partial \bar{\tau} = \partial/\partial \tau$ . It follows by applying these rules for the derivatives to Eq. (2.8) and then using (2.44) that

$$\bar{u}_{\bar{y}\bar{\tau}} = \frac{\bar{u}}{\bar{r}}. \quad (2.45)$$

This equation coincides with the SP equation (2.42) transformed by the hodograph transformation (2.43).

## E. Soliton solutions

A few particular solutions of the modified SP equation such as soliton and breather have been found in Ref. 13. For completeness, we present the one-soliton solution in the framework of our approach and investigate its property.

First, by solving the bilinear equations (2.16) and (2.17), we obtain the tau-functions for the one-soliton solution

$$f = 1 + e^{2\xi}, \quad g = \frac{2}{p} e^{\xi}, \quad \xi = py + \frac{1}{p} \tau + \xi_0, \quad (2.46)$$

where  $p$  and  $\xi_0$  are arbitrary real constants. The parametric representation of the one-soliton solution follows from (2.9) and (2.18). We write it in a convenient form in the following analysis:

$$u = \frac{1}{p} \operatorname{sech} \xi, \quad (2.47a)$$

$$X \equiv x + ct - x_0 = \frac{\xi}{p} - \frac{1}{p} \tanh \xi, \quad (2.47b)$$

where  $c = 1/p^2$  and  $x_0 = -(\xi_0 + 1)/p$ . This represents a localized pulse with the amplitude  $1/p$  moving to the left at the constant velocity  $c$ . The  $X$  derivative of  $u$  can be computed from (2.47) to give

$$u_X = \frac{u_\xi}{X_\xi} = -\frac{1}{\sinh \xi}. \quad (2.48)$$

It turns out from (2.48) that  $\lim_{X \rightarrow \pm 0} u_X = \mp \infty$ . To examine the profile of the pulse at the crest  $X = 0$  (or  $\xi = 0$ ), we expand  $u$  and  $X$  near the crest and obtain their leading-order asymptotics

$$u = \frac{1}{p} \left( 1 - \frac{\xi^2}{2} + O(\xi^4) \right), \quad X = \frac{1}{p} \left( \frac{\xi^3}{3} + O(\xi^5) \right). \quad (2.49)$$

Eliminating the variable  $\xi$  from (2.49) yields the profile of  $u$  near the crest

$$u = \frac{1}{p} \left( 1 - \frac{1}{2} (3pX)^{2/3} + O(X^{5/3}) \right). \quad (2.50)$$

Thus, the first derivative of  $u$  with respect to  $X$  does not exist at the crest, showing that  $u$  takes the form of a cusp soliton, as already observed in Ref. 13. This intriguing feature is in striking contrast to that of the SP equation for which its one-soliton solution is a loop soliton.

### III. Multi-component generalization

#### A. Multi-component bilinear system

To construct a multi-component analog of the modified SP equation, we start from the corresponding bilinear equations and their parametric solutions. Specifically, we generalize the parametric representation (2.9) and (2.18) in the form

$$u_i = \frac{g_i}{f}, \quad (i = 1, 2, \dots, n), \quad (3.1a)$$

$$x = y - (\ln f)_\tau, \quad (3.1b)$$

where the tau-functions  $f$  and  $g_i$  are assumed to satisfy the system of bilinear equations

$$D_y D_\tau g_i \cdot f = f g_i, \quad (i = 1, 2, \dots, n), \quad (3.2)$$

$$D_\tau^2 f \cdot f = \sum_{1 \leq j, k \leq n} c_{jk} g_j g_k, \quad (3.3)$$

which are the multi-component generalizations of (2.16) and (2.17), respectively. Here, the coefficients  $c_{jk}$  are the same as those introduced in (1.3b).

## B. Multi-component modified short pulse equations

The next step is to rewrite (3.2) and (3.3) in terms of the variables  $x$  and  $t$ . As will be demonstrated below, this can be achieved by means of the hodograph transformation

$$dy = rdx + rFdt, \quad d\tau = dt, \quad (3.4a)$$

or

$$\frac{\partial}{\partial y} = \frac{1}{r} \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} - F \frac{\partial}{\partial x}. \quad (3.4b)$$

where  $F$  is given by (1.3b). It turns out from (3.4) that the variable  $x = x(y, \tau)$  satisfies the system of linear PDEs

$$x_y = \frac{1}{r}, \quad x_\tau = -F. \quad (3.5)$$

The solvability condition of the system (3.5) yields the evolution equation for  $r$

$$r_\tau = r^2 F_y. \quad (3.6)$$

Now, we differentiate (3.1a) with respect to  $y$  and  $\tau$  and use (3.1a) and (3.2) to derive the relation

$$\left( \frac{g_i}{f} \right)_{y\tau} = \frac{D_y D_\tau g_i \cdot f}{f^2} - \frac{(D_y D_\tau f \cdot f) g_i}{f^3} = u_i - \frac{D_y D_\tau f \cdot f}{f^2} u_i, \quad (i = 1, 2, \dots, n). \quad (3.7)$$

We substitute (3.1b) into the first equation in (3.5) to deduce it into the form

$$\frac{D_y D_\tau f \cdot f}{f^2} = 2 \left( 1 - \frac{1}{r} \right). \quad (3.8)$$

Upon introducing (3.1a) and (3.8) into (3.7), we find that Eqs. (3.7) recast to

$$u_{i,y\tau} = \left( \frac{2}{r} - 1 \right) u_i, \quad (i = 1, 2, \dots, n). \quad (3.9)$$

The above system of equations is a multi-component analog of Eq. (2.8). We rewrite (3.9) in terms of the original variables  $x$  and  $t$  by using (3.4b)

$$u_{i,xt} = (F u_{i,x})_x + (2 - r) u_i, \quad (i = 1, 2, \dots, n). \quad (3.10)$$

The last step is to determine the functional form of  $r$ . To this end, we use  $F$  from (1.3b), (3.6) and (3.9) to derive the relation

$$\left( \frac{1}{r} \right)_\tau = -\frac{r}{2(2-r)} \sum_{1 \leq j, k \leq n} c_{jk} (u_{j,y} u_{k,y})_\tau. \quad (3.11)$$

Integrating (3.11) with respect to  $\tau$  under the boundary conditions  $u_{j,y} \rightarrow 0$  for  $j = 1, 2, \dots, n$  and  $r \rightarrow 1$  as  $|y| \rightarrow \infty$ , we obtain

$$\frac{1}{r} - \frac{1}{r^2} = \frac{1}{2} \sum_{1 \leq j, k \leq n} c_{jk} u_{j,y} u_{k,y}. \quad (3.12)$$

In view of the relations  $u_{j,y} = u_{j,x}/r$ ,  $u_{k,y} = u_{k,x}/r$  which stem from (3.4b), (3.12) becomes

$$r = 1 + \frac{1}{2} \sum_{1 \leq j, k \leq n} c_{jk} u_{j,x} u_{k,x}. \quad (3.13)$$

Upon substituting (1.3b) and (3.13) into (3.10), we arrive at the following multi-component system which is a generalization of the modified SP equation:

$$u_{i,xt} = u_i + \frac{1}{2} \left[ \left( \sum_{1 \leq j, k \leq n} c_{jk} u_j u_k \right) u_{i,x} \right]_x - \frac{1}{2} \left( \sum_{1 \leq j, k \leq n} c_{jk} u_{j,x} u_{k,x} \right) u_i, \quad (i = 1, 2, \dots, n). \quad (3.14)$$

### C. Remarks

**1.** The system of bilinear equations (3.2) and (3.3) coincides with that of the multi-component generalization of the SP equation proposed in Ref. 11 if one replaces the tau-functions  $g_i$  by  $2g_i$  ( $i = 1, 2, \dots, n$ ). In accordance with this observation, the former system is found to exhibit the  $N$ -soliton solution for the following cases: 1)  $c_{j,k} \neq 0$  ( $j \neq k$ ),  $c_{jj} = 0$ , ( $j, k = 1, 2, \dots, n$ ),<sup>11</sup> 2)  $c_{11} = c_{22} = 1$ ,  $c_{12} = c_{21} = 0$ ,<sup>11,17,18</sup> 3)  $c_{jj} = 1$ ,  $c_{jk} = 0$  ( $j \neq k$ ;  $j, k = 1, 2, 3, 4$ ).<sup>18</sup>

**2.** Solving (3.12) for  $1/r$ , we find

$$\frac{1}{r} = \frac{1}{2} \left( 1 \pm \sqrt{1 - 2 \sum_{1 \leq j, k \leq n} c_{jk} u_{j,y} u_{k,y}} \right). \quad (3.15)$$

Introducing this expression into (3.9) while taking into account the boundary condition  $r \rightarrow 1$ ,  $|y| \rightarrow \infty$ , we obtain a closed system of PDEs for  $u_i$

$$u_{i,y\tau} = \sqrt{1 - 2 \sum_{1 \leq j, k \leq n} c_{jk} u_{j,y} u_{k,y}} u_i, \quad (i = 1, 2, \dots, n). \quad (3.16)$$

**3.** By means of the transformations

$$\bar{u}_i = 2u_i, \quad (i = 1, 2, \dots, n), \quad \bar{x} = x - \int_{-\infty}^x (r - 1) dx, \quad \bar{t} = t, \quad (3.17)$$

we can derive the relation

$$\bar{u}_{i,\bar{x}\bar{t}} - \bar{u}_i - \frac{1}{2} (\bar{F} \bar{u}_{i,\bar{x}})_{\bar{x}} = \frac{2}{2-r} \{ u_{i,xt} - u_i - (F u_{i,x})_x - (1-r) u_i \}, \quad (i = 1, 2, \dots, n), \quad (3.18)$$

which is a multi-component analog of (2.41), where

$$\bar{F} = \frac{1}{2} \sum_{1 \leq j, k \leq n} c_{jk} \bar{u}_j \bar{u}_k. \quad (3.19)$$

Invoking Eqs. (3.14), we see that  $\bar{u}_i$  satisfy the system of PDEs

$$\bar{u}_{i, \bar{x} \bar{t}} = \bar{u}_i + \frac{1}{2} (\bar{F} \bar{u}_{i, \bar{x}})_{\bar{x}}, \quad (i = 1, 2, \dots, n). \quad (3.20)$$

This system is a multi-component generalization of the SP equation proposed in Ref. 11. We can also derive the relation (2.44) between  $r$  and  $\bar{r}$ , where  $r$  is given by (3.13) and  $\bar{r}$  by

$$\bar{r} = \sqrt{1 + \frac{1}{2} \sum_{1 \leq j, k \leq n} c_{jk} \bar{u}_{j, \bar{x}} \bar{u}_{k, \bar{x}}}. \quad (3.21)$$

The hodograph transformation (2.43) with  $\bar{r}$  from (3.21) and (3.4) recasts (3.9) to the system of PDEs

$$\bar{u}_{i, \bar{y} \bar{\tau}} = \frac{\bar{u}_i}{\bar{r}}, \quad (i = 1, 2, \dots, n), \quad (3.22)$$

which is a multi-component generalization of Eq. (2.45).

**4.** If we put  $c_{jj=1}$  ( $j = 1, 2, \dots, n$ ),  $c_{jk} = 0$  ( $j \neq k; j, k = 1, 2, \dots, n$ ) in Eq. (3.14) and take the continuum limit  $n \rightarrow \infty$ , then we obtain a (2+1)-dimensional nonlocal PDE

$$u_{xt} = u + \frac{1}{2} \left( u_x \int_{-\infty}^{\infty} u^2 dz \right)_x - \frac{1}{2} u \int_{-\infty}^{\infty} u_x^2 dz, \quad u = u(x, z, t). \quad (3.23)$$

The parametric representation for the solution of Eq. (3.23) can be expressed in the form

$$u = \frac{g}{f}, \quad x = y - \frac{f_\tau}{f}, \quad (3.24)$$

where the tau-functions  $f = f(y, \tau)$  and  $g = g(y, z, \tau)$  satisfy the system of bilinear equations

$$D_y D_\tau f \cdot g = fg, \quad D_\tau^2 f \cdot f = \int_{-\infty}^{\infty} g^2 dz. \quad (3.25)$$

The variables  $y$  and  $\tau$  in (3.24) and (3.25) are related to the original variables  $x$  and  $t$  by the hodograph transformation

$$dy = r dx + \frac{1}{2} \left( \int_{-\infty}^{\infty} u^2 dz \right) r dt, \quad d\tau = dt, \quad (3.26a)$$

where  $r = r(x, t)$  is given by

$$r = 1 + \frac{1}{2} \int_{-\infty}^{\infty} u_x^2 dz. \quad (3.26b)$$

The integrable feature of Eq. (3.23) will be discussed elsewhere.

#### IV. Two-component system

In this section, we consider the two-component system (1.8) which is a special case of (3.14) with  $n = 2$  and  $c_{11} = c_{22} = 0, c_{12} = 1$ . We give the Lax pair, an infinite number of conservation laws and multisoliton solutions for the system, establishing its integrability.

##### A. Integrability

For the two-component system, Eqs. (3.5) and (3.9) become

$$x_{y\tau} = -(uv)_y, \quad u_{y\tau} = (2x_y - 1)u, \quad v_{y\tau} = (2x_y - 1)v, \quad (4.1)$$

with the identifications  $u_1 = u, u_2 = v$  and  $F = uv$ . In these expressions,  $x_y = 1/r$ , where  $r$  is determined by the relation

$$\frac{1}{r} - \frac{1}{r^2} = u_y v_y. \quad (4.2)$$

Note from (3.13) with  $n = 2$  that  $r$  is expressed in the  $(x, t)$  coordinate system as

$$r = 1 + u_x v_x. \quad (4.3)$$

By the reduction  $u = v$ , Eqs. (4.1) reduce to Eqs. (2.2) and (2.8) whereas the expressions (4.2) and (4.3) reduce to (2.6) and (2.7), respectively. We found that the system of equations (4.1) admits the following Lax pair

$$\Psi_y = U\Psi, \quad \Psi_\tau = V\Psi, \quad (4.4a)$$

with

$$U = \lambda \begin{pmatrix} 2x_y - 1 & 2u_y \\ 2v_y & -(2x_y - 1) \end{pmatrix}, \quad V = \begin{pmatrix} \frac{1}{4\lambda} & -u \\ v & -\frac{1}{4\lambda} \end{pmatrix}, \quad (4.4b)$$

where  $\lambda$  is a spectral parameter. Indeed, it follows from the compatibility condition  $\Psi_{y\tau} = \Psi_{\tau y}$  that

$$U_\tau - V_y + UV - VU = O, \quad (4.5)$$

which yields Eqs. (4.1).

The Lax pair associated with the system (1.8) is derived by rewriting (4.4) in terms of the variables  $x$  and  $t$ . This can be attained simply by applying the hodograph transformation (3.4) with  $F = uv$  and  $r$  from (4.3) to (4.4), giving

$$\Psi_x = \tilde{U}\Psi, \quad \Psi_t = \tilde{V}\Psi, \quad (4.6a)$$

with

$$\tilde{U} = \lambda \begin{pmatrix} 2 - r & 2u_x \\ 2v_x & -(2 - r) \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} \frac{1}{4\lambda} + \lambda(2 - r)uv & -u + 2\lambda uv u_x \\ v + 2\lambda uv v_x & -\frac{1}{4\lambda} - \lambda(2 - r)uv \end{pmatrix}. \quad (4.6b)$$

It is easy to confirm that the compatibility condition of the Lax pair (4.6) yields the system of equations (1.8).

## B. Conservation laws

A striking feature of the completely integrable system is the existence of an infinite number of conservation laws. Several practical methods are now available for deriving the conservation laws. Among them, we employ a procedure based on the Lax pair. See Ref. 19, for example.

Let  $\Psi = (\psi_1, \psi_2)^T$ ,  $\tilde{U} = (u_{ij})_{1 \leq i, j \leq 2}$ ,  $\tilde{V} = (v_{ij})_{1 \leq i, j \leq 2}$ . Then, the Lax pair (4.6) is written in terms of the components as

$$\psi_{1,x} = u_{11}\psi_1 + u_{12}\psi_2, \quad \psi_{1,t} = v_{11}\psi_1 + v_{12}\psi_2, \quad (4.7a)$$

$$\psi_{2,x} = u_{21}\psi_1 + u_{22}\psi_2, \quad \psi_{2,t} = v_{21}\psi_1 + v_{22}\psi_2. \quad (4.7b)$$

Introducing the variables  $\Gamma = \psi_2/\psi_1$  and  $\bar{\Gamma} = \psi_1/\psi_2$ , we can recast (4.7) to

$$(\ln \psi_1)_x = u_{11} + u_{12}\Gamma, \quad (\ln \psi_1)_t = v_{11} + v_{12}\Gamma, \quad (4.8a)$$

$$(\ln \psi_2)_x = u_{22} + u_{21}\bar{\Gamma}, \quad (\ln \psi_2)_t = v_{22} + v_{21}\bar{\Gamma}. \quad (4.8b)$$

It follows from the compatibility conditions of (4.8) that

$$(u_{11} + u_{12}\Gamma)_t = (v_{11} + v_{12}\Gamma)_x, \quad (4.9a)$$

$$(u_{22} + u_{21}\bar{\Gamma})_t = (v_{22} + v_{21}\bar{\Gamma})_x. \quad (4.9b)$$

Integrating (4.9) with respect to  $x$ , we can see that the quantities  $\int_{-\infty}^{\infty} (u_{11} + u_{12}\Gamma)dx$ ,  $\int_{-\infty}^{\infty} (u_{22} + u_{21}\bar{\Gamma})dx$  are conserved in time. We then use (4.7) to derive the relations

$$(u_{12}\Gamma)_x = u_{12}u_{21} + u_{12,x}\Gamma + (u_{22} - u_{11})u_{12}\Gamma - (u_{12}\Gamma)^2, \quad (4.10a)$$

$$(u_{21}\bar{\Gamma})_x = u_{12}u_{21} + u_{21,x}\bar{\Gamma} + (u_{11} - u_{22})u_{21}\bar{\Gamma} - (u_{21}\bar{\Gamma})^2, \quad (4.10b)$$

which reduce, after substituting the components of the matrix  $\tilde{U}$ , to

$$(u_{12}\Gamma)_x = 4\lambda^2 u_x v_x + \frac{u_{xx}}{u_x} (u_{12}\Gamma) - 2\lambda(2-r)(u_{12}\Gamma) - (u_{12}\Gamma)^2, \quad (4.11a)$$

$$(u_{21}\bar{\Gamma})_x = 4\lambda^2 u_x v_x + \frac{v_{xx}}{v_x} (u_{21}\bar{\Gamma}) + 2\lambda(2-r)(u_{21}\bar{\Gamma}) - (u_{21}\bar{\Gamma})^2. \quad (4.11b)$$

Note that (4.11a) transforms to (4.11b) by interchanging the variables  $u$  and  $v$  and replacing  $\lambda$  by  $-\lambda$ . This reflects the invariance of the system (1.8) under the interchange of the variables  $u$  and  $v$ . Thus, we may use either (4.11a) or (4.11b) to obtain conservation laws.

Let us now derive the conservation laws. We recall that  $\int_{-\infty}^{\infty} (u_{11} + u_{12}\Gamma)_t dx = 0$ . However, since  $\int_{-\infty}^{\infty} u_{11,t} dx = 0$ , as confirmed easily,  $\int_{-\infty}^{\infty} u_{12}\Gamma dx$  is a conserved quantity.

We expand  $u_{12}\Gamma$  in powers of  $\lambda$  and substitute it into (4.11a). We find two such expansions which will lead to both the local and nonlocal conserved quantities. We consider the two cases separately.

### 1. Local conservation laws

An expansion which yields the local conservation laws, i.e., integrals of the variables  $u$  and  $v$  and their  $x$ -derivatives, is given by

$$u_{12}\Gamma = \sum_{n=0}^{\infty} \gamma_n \lambda^{1-n}. \quad (4.12)$$

We impose the boundary conditions  $u, v, u_x, v_x, \dots, \gamma_n (n = 0, 1, 2, \dots) \rightarrow 0$  as  $|x| \rightarrow \infty$  to assure the convergence of the integrals associated with the conservation laws. Substituting (4.12) into (4.11a) and comparing the coefficients of  $\lambda^{1-n}$  on both sides, we obtain the recursion relation that determines  $\gamma_n$

$$\gamma_{n,x} = \frac{u_{xx}}{u_x} \gamma_n - 2(2 - r + \gamma_0)\gamma_{n+1} - \sum_{m=1}^n \gamma_{n-m+1}\gamma_m, \quad (n \geq 1), \quad (4.13)$$

where  $\gamma_0$  satisfies the quadratic equation

$$\gamma_0^2 + 2(2 - r)\gamma_0 - 4u_x v_x = 0, \quad (4.14)$$

which stems from the coefficients of  $\lambda^2$ . It follows from the coefficients of order  $\lambda$  that

$$\gamma_{0,x} = \frac{u_{xx}}{u_x} \gamma_0 - 2(2 - r + \gamma_0)\gamma_1. \quad (4.15)$$

With  $r$  from (4.3), we find two solutions of Eq. (4.14),  $\gamma_0 = 2u_x v_x$  and  $\gamma_0 = -2$ . We substitute the former solution into (4.15) to obtain

$$\gamma_1 = -\frac{u_x v_{xx}}{1 + u_x v_x}. \quad (4.16)$$

Note that the solution  $\gamma_0 = -2$  is irrelevant since it does not satisfy the boundary condition. We put (4.13) in a form which is suitable for determining  $\gamma_n$  successively

$$\gamma_{n+1} = \frac{1}{2(1 + u_x v_x)} \left( -\gamma_{n,x} + \frac{u_{xx}}{u_x} \gamma_n - \sum_{m=1}^n \gamma_{n-m+1}\gamma_m \right), \quad (n \geq 1). \quad (4.17)$$

The  $n$ th conservation law which is denoted by  $I_n$  is given by

$$I_n = \int_{-\infty}^{\infty} \gamma_n dx, \quad (n \geq 0). \quad (4.18)$$



The recursion relation (4.17) can be solved successively starting with the initial condition (4.16), the first two of which read

$$\gamma_2 = -\frac{1}{2} \frac{u_{xx}v_{xx}}{(1+u_xv_x)^3} + \frac{1}{2} \left( \frac{u_xv_{xx}}{(1+u_xv_x)^2} \right)_x, \quad (4.19a)$$

$$\gamma_3 = \frac{1}{2(1+u_xv_x)} \left\{ -\gamma_{2,x} + \left( \frac{u_{xx}}{u_x} + \frac{2u_xv_{xx}}{1+u_xv_x} \right) \gamma_2 \right\}. \quad (4.19b)$$

Although the computation of  $\gamma_n$  becomes very complicated as  $n$  increases, it can be continued to generate explicit expressions of  $\gamma_n$ . An inspection of the structure of (4.17) reveals that the resulting conservation laws have local character. Thus, up to overall constants, the first four conservation laws are found to be as follows:

$$I_0 = \int_{-\infty}^{\infty} u_x v_x dx, \quad (4.20a)$$

$$I_1 = \int_{-\infty}^{\infty} \frac{1}{r} (u_x v_{xx} - u_{xx} v_x) dx, \quad (4.20b)$$

$$I_2 = \int_{-\infty}^{\infty} \frac{u_{xx} v_{xx}}{r^3} dx, \quad (4.20c)$$

$$I_3 = \int_{-\infty}^{\infty} \left[ \frac{1}{r^4} (u_{xx} v_{xxx} - u_{xxx} v_{xx}) - \frac{2}{r^5} u_{xx} v_{xx} (u_x v_{xx} - u_{xx} v_x) \right] dx. \quad (4.20d)$$

In deriving these formulas, we have used the fact that if  $I_n$  is a conserved quantity, then the quantity  $\tilde{I}_n$  which is obtained from  $I_n$  by interchanging the variables  $u$  and  $v$  is also conserved. Consequently,  $I_n \pm \tilde{I}_n$  become the conservation laws as well. Note that  $I_0$  and  $I_2$  are symmetric and  $I_1$  and  $I_3$  are antisymmetric with respect to  $u$  and  $v$ , respectively.

## 2. Nonlocal conservation laws

An expansion of  $u_{12}\Gamma$  exists which produces nonlocal conservation laws. To be more specific, we expand  $u_{12}\Gamma$  in powers of  $\lambda$  as

$$u_{12}\Gamma = \sum_{n=1}^{\infty} \bar{\gamma}_n \lambda^n. \quad (4.21)$$

Substituting (4.21) into (4.11a) and comparing the coefficient of  $\lambda^n$  on both sides, we obtain the recursion relation that determines  $\bar{\gamma}_n$

$$\bar{\gamma}_1 = u_x, \quad \bar{\gamma}_{n,x} = 4u_x v_x \delta_{n,2} + \frac{u_{xx}}{u_x} \bar{\gamma}_n - 2(2-r) \bar{\gamma}_{n-1} - \sum_{m=1}^{n-1} \bar{\gamma}_{n-m} \bar{\gamma}_m, \quad (n \geq 2), \quad (4.22)$$

where  $\delta_{n,2}$  is Kronecker's delta. To solve the recursion relation for  $\bar{\gamma}_n$ , it is suitable to introduce the quantity  $\hat{\gamma}_n$  by  $\bar{\gamma}_n = u_x \hat{\gamma}_n$ . Thus, (4.22) recasts to

$$\hat{\gamma}_1 = 1, \quad \hat{\gamma}_{n,x} = 4v_x \delta_{n,2} - 2(2-r)\hat{\gamma}_{n-1} - u_x \sum_{m=1}^{n-1} \hat{\gamma}_{n-m} \hat{\gamma}_m, \quad (n \geq 2). \quad (4.23)$$

This recursion relation can be solved successively with the initial condition  $\hat{\gamma}_1 = 1$ . Then, the  $n$ th conservation law  $J_n$  is given by the formula

$$J_n = \int_{-\infty}^{\infty} \bar{\gamma}_n dx = - \int_{-\infty}^{\infty} u \hat{\gamma}_{n,x} dx, \quad (n = 1, 2, \dots), \quad (4.24)$$

where the last line follows by the integration by parts.

Before proceeding, we derive the useful relations which are a consequence of the system (1.8). Specifically, we introduce the new variables  $p$  and  $q$  according to  $p = u(1 - u_x v_x)$  and  $q = v(1 - u_x v_x)$  and put (1.8) into the form

$$u_{xt} = p + (uvu_x)_x \quad v_{xt} = q + (uvv_x)_x. \quad (4.25)$$

It immediately follows from (4.25) and the boundary conditions  $u, v, u_x, v_x \rightarrow 0, |x| \rightarrow \infty$  that

$$\int_{-\infty}^{\infty} p dx = \int_{-\infty}^{\infty} u(1 - u_x v_x) dx = 0, \quad \int_{-\infty}^{\infty} q dx = \int_{-\infty}^{\infty} v(1 - u_x v_x) dx = 0, \quad (4.26)$$

and hence

$$\int_{-\infty}^{\infty} p_t dx = 0, \quad \int_{-\infty}^{\infty} q_t dx = 0. \quad (4.27)$$

We use Eqs. (4.25) integrated once with respect to  $x$  to derive the evolution equations of  $p$  and  $q$ . They read

$$p_t = (1 - u_x v_x) \partial_x^{-1} p + 2uvu_x - [u^2 v(1 + u_x v_x)]_x, \quad (4.28a)$$

$$q_t = (1 - u_x v_x) \partial_x^{-1} q + 2uvv_x - [uv^2(1 + u_x v_x)]_x, \quad (4.28b)$$

where  $\partial_x^{-1} = \int_{-\infty}^x dy$  is an integral operator. If we substitute (4.28) into (4.27), we obtain the relations

$$\int_{-\infty}^{\infty} [(1 - u_x v_x) \partial_x^{-1} p + 2uvu_x] dx = 0, \quad \int_{-\infty}^{\infty} [(1 - u_x v_x) \partial_x^{-1} q + 2uvv_x] dx = 0. \quad (4.29)$$

Now, the recursion relation (4.23) yields the formulas

$$\hat{\gamma}_{2,x} = 4v_x - 2(2-r)\hat{\gamma}_1 - u_x, \quad (4.30a)$$

$$\hat{\gamma}_{3,x} = -2(2-r)\hat{\gamma}_2 - 2u_x \hat{\gamma}_2, \quad (4.30b)$$

$$\hat{\gamma}_{4,x} = -2(2-r)\hat{\gamma}_3 - 2u_x\hat{\gamma}_3 - u_x\hat{\gamma}_2^2, \quad (4.30c)$$

$$\hat{\gamma}_{5,x} = -2(2-r)\hat{\gamma}_4 - 2u_x\hat{\gamma}_4 - 2u_x\hat{\gamma}_2\hat{\gamma}_3. \quad (4.30d)$$

It is straightforward to obtain the conservation laws by substituting (4.30) into (4.24) and using the relations (4.26) and (4.29). We omit the detail of the calculations and write only the final results. Up to overall constants, the first five conservation laws read

$$J_1 = 0, \quad (4.31a)$$

$$J_2 = \int_{-\infty}^{\infty} (uv_x - u_xv)dx, \quad (4.31b)$$

$$J_3 = \int_{-\infty}^{\infty} (1 - u_xv_x)uvdx, \quad (4.31c)$$

$$J_4 = \int_{-\infty}^{\infty} \left[ u^2vv_x - uu_xv^2 + (1 - u_xv_x)u\partial_x^{-1}\{(1 - u_xv_x)v - (1 - u_xv_x)v\partial_x^{-1}\{(1 - u_xv_x)u\}\} \right] dx, \quad (4.31d)$$

$$J_5 = \int_{-\infty}^{\infty} \left[ (1 - u_xv_x)u^2v^2 + 2uvu_x\partial_x^{-1}\{(1 - u_xv_x)v\} + 2uvv_x\partial_x^{-1}\{(1 - u_xv_x)u\} + (1 - u_xv_x)\partial_x^{-1}\{(1 - u_xv_x)u\}\partial_x^{-1}\{(1 - u_xv_x)v\} \right] dx. \quad (4.31e)$$

We can see that the conservation laws  $J_n$  for  $n \geq 4$  become nonlocal in view of the presence of the integral operator  $\partial_x^{-1}$ .

Last, we conclude this section with a few comments. First, we observe that under the reduction  $u = v$ , the conservation laws reduce to those of the modified SP equation (1.6). Actually, the nontrivial conservation laws follows from (4.20) and (4.31):

$$I_0 = \int_{-\infty}^{\infty} u_x^2 dx, \quad (4.32a)$$

$$I_2 = \int_{-\infty}^{\infty} \frac{u_{xx}^2}{(1 + u_x^2)^3} dx, \quad (4.32b)$$

$$J_3 = \int_{-\infty}^{\infty} (1 - u_x^2)u^2 dx, \quad (4.33a)$$

$$J_5 = \int_{-\infty}^{\infty} \left[ -\frac{1}{3}(1 - u_x^2)u^4 + (1 - u_x^2) \{ \partial_x^{-1}(1 - u_x^2)u \}^2 \right] dx. \quad (4.33b)$$

The second comment is concerned with the conservation laws of the two-component SP equation (1.2). As already shown in Sec. III, the transformations

$$\bar{u} = 2u, \quad \bar{v} = 2v, \quad \bar{x} = x - \int_{-\infty}^x u_xv_x dx, \quad \bar{t} = t, \quad (4.34)$$

lead to (1.2) when applied to the two-component modified SP equation (1.8). This fact enables us to derive the conservation laws of the system (1.2) in a simple manner. Indeed, using the relations  $u_x = (1/2)(1 - u_x v_x) \bar{u}_x$ ,  $v_x = (1/2)(1 - u_x v_x) \bar{v}_x$  and the two-component analog of (2.44), the conservation laws  $I_n$  and  $J_n$  from (4.20) and (4.31) reduce, up to overall constants, to  $\bar{I}_n$  and  $\bar{J}_n$ , respectively, where

$$\bar{I}_0 = \int_{-\infty}^{\infty} (\bar{r} - 1) dx, \quad (4.35a)$$

$$\bar{I}_1 = \int_{-\infty}^{\infty} \frac{1}{\bar{r}(\bar{r} + 1)} (u_x v_{xx} - u_{xx} v_x) dx, \quad (4.35b)$$

$$\bar{I}_2 = \int_{-\infty}^{\infty} \left[ \frac{u_{xx} v_{xx}}{\bar{r}^3} - \frac{\{(u_x v_x)_x\}^2}{4\bar{r}^5} \right] dx, \quad (4.35c)$$

$$\bar{I}_3 = \int_{-\infty}^{\infty} \frac{u_{xx} v_{xxx} - u_{xxx} v_{xx}}{\bar{r}^5} dx, \quad (4.35d)$$

$$\bar{J}_2 = \int_{-\infty}^{\infty} (u v_x - u_x v) dx, \quad (4.36a)$$

$$\bar{J}_3 = \int_{-\infty}^{\infty} u v dx, \quad (4.36b)$$

$$\bar{J}_4 = \int_{-\infty}^{\infty} [u^2 v v_x - u u_x v^2 + 4u \partial_x^{-1} v - 4v \partial_x^{-1} u] dx, \quad (4.36c)$$

$$\bar{J}_5 = \int_{-\infty}^{\infty} [u^2 v^2 + 2uv(u_x \partial_x^{-1} v + v_x \partial_x^{-1} u) + 4(\partial_x^{-1} u)(\partial_x^{-1} v)] dx. \quad (4.36d)$$

Here,  $\bar{r} = \sqrt{1 + u_x v_x}$ , and the overbar attached to the variables  $u, v$  and  $x$  has been deleted for simplicity. Performing the reduction  $u = v$  in the above expressions, we recover the conservation laws of the SP equation.<sup>6,7</sup>

### C. Soliton solutions

The soliton solutions of the two-component modified SP equation are constructed by solving the system of bilinear equations (3.2) and (3.3) with  $n = 2$  and  $c_{11} = c_{22} = 0, c_{12} = 1$ . As already noticed, this system has the same form as that of the two-component SP equation (2.2). For the latter system, the tau-functions  $f, g_1$  and  $g_2$  for the  $N$ -soliton solution have already been presented. See expressions (4.6) in Ref. 11. The only difference in the parametric representation of the solution lies in the coordinate transformation. Specifically, the expression corresponding to (2.18) takes the form  $x = y - 2(\ln f)_\tau$  for the two-component SP equation. We can construct multisoliton solutions in which each component contains arbitrary number of solitons. Here, we restrict our consideration to the case where both  $u$  and  $v$  contain  $N$  solitons. We discuss the properties of the one- and two-soliton solutions, as well as the breather solution. The general  $N$ -soliton solution will be addressed shortly.

## 1. One-soliton solution

The tau-functions for the one-soliton solution are given by

$$f = 1 + \frac{a_1 b_1 p_1^2}{4} z_1^2, \quad g_1 = a_1 z_1, \quad g_2 = b_1 z_1, \quad (4.37a)$$

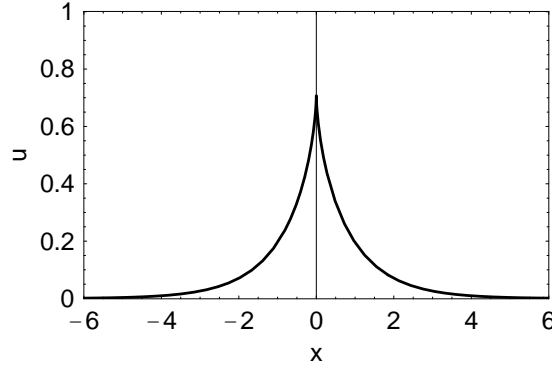
$$z_1 = e^{\xi_1}, \quad \xi_1 = p_1 y + \frac{1}{p_1} \tau + \xi_{10}, \quad (4.37b)$$

where  $a_1, a_2$  and  $p_1$  are real constants which are assumed to be positive here, and  $\xi_{10}$  is the phase constant. The parametric soliton solution is calculated from (3.1) to give

$$u = \frac{1}{p_1} \sqrt{\frac{a_1}{b_1}} \operatorname{sech}(\xi_1 + \delta_1), \quad v = \frac{1}{p_1} \sqrt{\frac{b_1}{a_1}} \operatorname{sech}(\xi_1 + \delta_1), \quad (4.38a)$$

$$x = y - \frac{1}{p_1} \tanh(\xi_1 + \delta_1), \quad \delta_1 = \ln \left( \frac{\sqrt{a_1 b_1} p_1}{2} \right). \quad (4.38b)$$

The profile of  $u$  is depicted in Fig. 1. It represents a cusp soliton with the amplitude  $\frac{1}{p_1} \sqrt{\frac{a_1}{b_1}}$  and the velocity  $c_1 = 1/p_1^2$ . The property of  $v$  is the same as that of  $u$  except the amplitude and the velocity being given respectively by  $\frac{1}{p_1} \sqrt{\frac{b_1}{a_1}}$  and  $1/p_2^2$ . By comparing (2.47) and (4.38), we see that the cusp soliton has the same structure as that of the cusp soliton (2.47) of the modified SP equation.



**FIG. 1.** The profile of a cusp soliton solution  $u$  with the parameters  $p_1 = 1.0, a_1 = 0.5$  and  $b_1 = 1.0$ .

## 2. Two-soliton solution

The tau-functions for the two-soliton solution read

$$f = 1 + \frac{1}{4} a_1 b_1 p_1^2 z_1^2 + (a_1 b_2 + a_2 b_1) \frac{(p_1 p_2)^2}{(p_1 + p_2)^2} z_1 z_2 + \frac{1}{4} a_2 b_2 p_2^2 z_2^2$$

$$+\frac{1}{16}a_1a_2b_1b_2\frac{(p_1p_2)^2(p_1-p_2)^4}{(p_1+p_2)^4}(z_1z_2)^2, \quad (4.39a)$$

$$g_1 = a_1z_1 + a_2z_2 + \frac{1}{4}a_1a_2b_1\frac{p_1^2(p_1-p_2)^2}{(p_1+p_2)^2}z_1^2z_2 + \frac{1}{4}a_1a_2b_2\frac{p_2^2(p_1-p_2)^2}{(p_1+p_2)^2}z_1z_2^2, \quad (4.39b)$$

$$g_2 = b_1z_1 + b_2z_2 + \frac{1}{4}a_1b_1b_2\frac{p_1^2(p_1-p_2)^2}{(p_1+p_2)^2}z_1^2z_2 + \frac{1}{4}a_2b_1b_2\frac{p_2^2(p_1-p_2)^2}{(p_1+p_2)^2}z_1z_2^2. \quad (4.39c)$$

The asymptotic analysis of the two-soliton solution can be performed parallel to that of the two-soliton solution of the two-component SP equation. Hence, we summarize the results. We are concerned only with  $u$  since the corresponding asymptotic formulas for  $v$  follow by interchanging the parameters  $a_i$  and  $b_i$  ( $i = 1, 2$ ).

We decompose  $u$  as  $u = U_1 + U_2$ . Then, as  $t \rightarrow -\infty$ ,  $U_1$ ,  $U_2$  and  $x$  behave like

$$U_1 \sim \frac{1}{p_1}\sqrt{\frac{a_1}{b_1}}\operatorname{sech}(\xi_1 + \delta'_1), \quad (4.40a)$$

$$x + c_1t - x_{10} \sim \frac{\xi_1}{p_1} - \frac{1}{p_1}\tanh(\xi_1 + \delta'_1) - \frac{1}{p_1} - \frac{2}{p_2}, \quad (4.40b)$$

$$U_2 \sim \frac{1}{p_2}\sqrt{\frac{a_2}{b_2}}\operatorname{sech}(\xi_2 + \delta_2), \quad (4.41a)$$

$$x + c_2t - x_{20} \sim \frac{\xi_1}{p_2} - \frac{1}{p_2}\tanh(\xi_2 + \delta_2) - \frac{1}{p_2}, \quad (4.41b)$$

where

$$c_i = \frac{1}{p_i^2}, \quad \delta_i = \ln\left(\frac{\sqrt{a_i b_i}}{4}p_i\right), \quad \delta'_i = \ln\left[\frac{\sqrt{a_i b_i}}{2}p_i\left(\frac{p_1 - p_2}{p_1 + p_2}\right)^2\right], \quad (i = 1, 2). \quad (4.42)$$

As  $t \rightarrow +\infty$ , the corresponding asymptotic forms are given by

$$U_1 \sim \frac{1}{p_1}\sqrt{\frac{a_1}{b_1}}\operatorname{sech}(\xi_1 + \delta_1), \quad (4.43a)$$

$$x + c_1t - x_{10} \sim \frac{\xi_1}{p_1} - \frac{1}{p_1}\tanh(\xi_1 + \delta_1) - \frac{1}{p_1}, \quad (4.43b)$$

$$U_2 \sim \frac{1}{p_2}\sqrt{\frac{a_2}{b_2}}\operatorname{sech}(\xi_2 + \delta'_2), \quad (4.44a)$$

$$x + c_2t - x_{20} \sim \frac{\xi_1}{p_2} - \frac{1}{p_2}\tanh(\xi_2 + \delta'_2) - \frac{1}{p_2} - \frac{2}{p_1}. \quad (4.44b)$$

We can see that the asymptotic state of the solution for large time is represented by a superposition of two cusp solitons, each of which has the same form as that of the cusp

soliton solution given by (4.38). The net effect of the interaction is the phase shifts caused by the collision of solitons. Let  $\Delta_1$  and  $\Delta_2$  be the phase shifts of  $U_1$  and  $U_2$ , respectively which are defined by

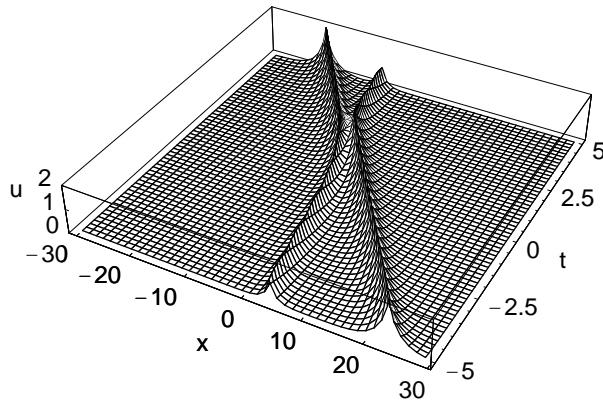
$$\Delta_i = x_{ic}(t \rightarrow -\infty) - x_{ic}(t \rightarrow +\infty), \quad (i = 1, 2). \quad (4.45)$$

where  $x_{ic}$  is the center position of the  $i$ th soliton. It follows from (4.40)-(4.44) that

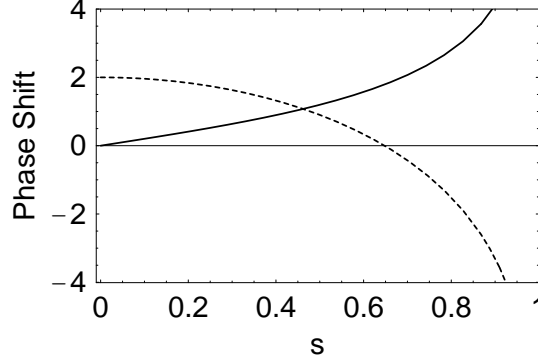
$$\Delta_1 = -\frac{1}{p_1} \ln \left( \frac{p_1 - p_2}{p_1 + p_2} \right)^2 - \frac{2}{p_2}, \quad (4.46a)$$

$$\Delta_2 = \frac{1}{p_2} \ln \left( \frac{p_1 - p_2}{p_1 + p_2} \right)^2 + \frac{2}{p_1}. \quad (4.46b)$$

The asymptotic amplitudes of  $U_1$  and  $U_2$  for large time are given respectively by  $A_1 = (1/p_1)\sqrt{a_1/b_1}$  and  $A_2 = (1/p_2)\sqrt{a_2/b_2}$  and the velocities by  $c_1 = 1/p_1^2$  and  $c_2 = 1/p_2^2$ , respectively. Suppose that  $A_1 > A_2$  and  $c_1 > c_2$  ( $p_1 < p_2$ ), implying that the large soliton travels faster than the small soliton. Figure 2 shows the interaction process of two cusp solitons for the component  $u$  with  $A_1 = 2, A_2 = 1, c_1 = 4$  and  $c_2 = 1$ . Figure 3 plots  $p_1\Delta_1$  and  $p_1\Delta_2$  as a function of  $s(=p_1/p_2)$ . We can see that the phase shift  $\Delta_1$  of the large soliton is always positive whereas the small soliton exhibits a positive phase shift for  $0 < s < s_c$  and a negative phase shift for  $s_c < s < 1$ , where  $s_c = 0.648$  is a solution of the transcendental equation  $\Delta_2 = 0$ . In the present example,  $\Delta_1 = 2.39$  and  $\Delta_2 = 1.80$ . Recall that this peculiar feature of the phase shift has been observed in the interaction process of loop solitons of the SP and two-component SP equations.<sup>9,11</sup> Another novel aspect of the solution is that the small soliton can travel faster than the large soliton if the condition  $1 < p_1/p_2 < \sqrt{a_1b_2/a_2b_1}$  is satisfied so that  $A_1 > A_2$  and  $c_1 < c_2$ .



**FIG. 2.** The interaction process of the two-cusp soliton solution  $u$  with the parameters  $p_1 = 0.5, p_2 = 1.0, a_1 = 1.0, a_2 = 2.0, b_1 = 1.0, b_2 = 2.0$  and  $\xi_{10} = \xi_{20} = 0$ .



**FIG. 3.** The phase shifts  $p_1\Delta_1$  and  $p_1\Delta_2$  as a function of  $s(=p_1/p_2)$ . The solid (broken) line represents the phase shift of the large (small) cusp soliton.

### 3. One-breather solution

The breather has a localized structure which oscillates with time and decays in space at infinity. Similar to the breather solution of the sine-Gordon equation which is the bound state of a kink and an antikink, the present two-component system supports breather solutions. Of particular interest is the one-breather solution which can be constructed from the two-soliton solution by means of a special parameterization. Actually, we put

$$p_1 = a + ib = p_2^*, \quad \xi_{10} = \lambda + i\mu = \xi_{20}^*, \quad a_1 = \alpha e^{i\phi} = a_2^*, \quad b_1 = \beta e^{i\psi} = b_2^*, \quad (4.47)$$

in (4.39) and obtain from (3.1) with  $n = 2$  the parametric representation of the solution

$$u = \frac{2ab\sqrt{\frac{\alpha}{\beta}}}{\sqrt{a^2 + b^2}} \frac{\hat{g}_1}{\hat{f}}, \quad v = \frac{2ab\sqrt{\frac{\beta}{\alpha}}}{\sqrt{a^2 + b^2}} \frac{\hat{g}_2}{\hat{f}}, \quad (4.48a)$$

$$x = y - \frac{ab}{a^2 + b^2} \frac{b \sinh 2\theta + a \sin 2\chi}{\hat{f}}, \quad (4.48b)$$

with

$$\hat{f} = b^2 \cosh^2 \theta + a^2 \cos^2 \chi - (a^2 + b^2) \sin^2 \delta, \quad (4.49a)$$

$$\hat{g}_1 = \sin(\chi_0 - \delta) \sin \chi \cosh \theta - \cos(\chi_0 - \delta) \cos \chi \sinh \theta, \quad (4.49b)$$

$$\hat{g}_2 = \sin(\chi_0 + \delta) \sin \chi \cosh \theta - \cos(\chi_0 + \delta) \cos \chi \sinh \theta, \quad (4.49c)$$

$$\theta = a \left( y + \frac{1}{a^2 + b^2} \tau \right) + \lambda, \quad \chi = b \left( y - \frac{1}{a^2 + b^2} \tau \right) + \mu, \quad (4.49d)$$

$$\tan \chi_0 = \frac{b}{a}, \quad \delta = \frac{1}{2}(\phi - \psi). \quad (4.49e)$$

Here,  $a, b, \alpha$  and  $\beta$  are positive constants,  $\lambda, \mu, \phi$  and  $\psi$  are real constants, the asterisk denotes complex conjugate, and the appropriate shifts of the variables  $x, y$  and  $\tau$  have

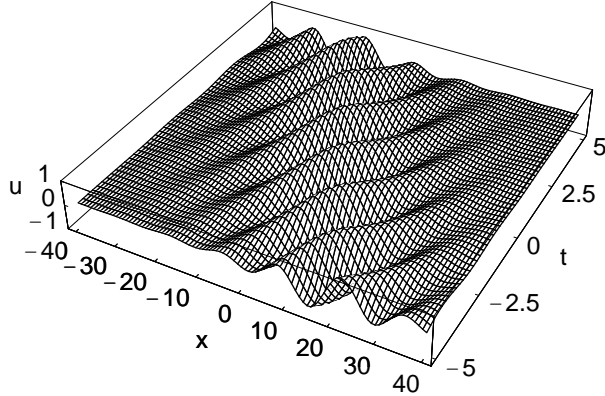


been performed to put the solution in a transparent form. Recall that the two-component modified SP equation (1.8) and its transformed form (4.1) via the hodograph transformation (3.4) are invariant under the scale transformation  $u \rightarrow u_0 u, v \rightarrow v_0 v, x \rightarrow b^{-1} x, y \rightarrow b^{-1} y, t \rightarrow b t, \tau \rightarrow b \tau$  if the condition  $u_0 v_0 b^2 = 1$  is imposed on the constants  $u_0, v_0$  and  $b$ . Applying this scale transformation to the solution (4.48) reveals that only the two parameters  $a/b$  and  $\delta$  characterize the solution. The parameters  $\lambda$  and  $\mu$  in (4.49d) can be set to zero thanks to the translational invariance of (1.8).

The breather solution presented here would exhibit singularities unless we impose certain condition on the parameters  $a, b$  and  $\delta$ . We shall address this issue. The smooth breather solution is produced if the inequalities  $\hat{f} > 0$  and  $x_y > 0$  hold for arbitrary values of  $\theta$  and  $\chi$ . The former condition is equivalent to requiring that the solution is finite in space and time, whereas the latter one is that the hodograph mapping (3.4) is one-to-one so that the solution becomes a single-valued function of  $x$  for arbitrary  $t$ . The detailed analysis shows that the former inequality turns out to be  $0 < a/b < 1/\sqrt{|\tan \delta|}$ , whereas the latter one leads to the inequality

$$0 < \frac{a}{b} < \sqrt{\frac{1 - |\sin \delta|}{1 + |\sin \delta|}}, \quad -\pi \leq \delta \leq \pi. \quad (4.50)$$

Since the permissible range of  $a/b$  from (4.50) is included in the range from the former inequality, we impose (4.50) for the smooth breather solution to exist. Figure 4 shows the time evolution of the one-breather solution  $u$  where the parameters are given by  $a = 0.1, b = 0.5, \alpha = \beta = 1, \delta = \pi/4$  ( $\phi = \pi/2, \psi = 0$ ).



**FIG. 4.** The time evolution of the one-breather solution  $u$ .

The properties of the solution depends critically on the parameter  $\delta$ . In particular, for  $\delta = 0$ , the parametric solution (4.48) takes the form

$$u = \frac{2ab\sqrt{\frac{\alpha}{\beta}}}{a^2 + b^2} \frac{b \sin \chi \cosh \theta - a \cos \chi \sinh \theta}{b^2 \cosh^2 \theta + a^2 \cos^2 \chi}, \quad v = \frac{\beta}{\alpha} u, \quad (4.51a)$$

$$x = y - \frac{ab}{a^2 + b^2} \frac{b \sinh 2\theta + a \sin 2\chi}{b^2 \cosh^2 \theta + a^2 \cos^2 \chi}. \quad (4.51b)$$

Under this special setting, the inequality (4.50) becomes  $0 < a/b < 1$ , and the parameter  $\chi_0$  from (4.49e) must satisfy the inequality  $\tan \chi_0 > 1$ . In the above expression, we have assumed  $\pi/4 < \chi_0 < \pi/2$  so that  $\cos \chi_0 > 0$ . One can see from (4.51) that the solution describes the propagation of linearly polarized waves. Note that if  $\alpha = \beta$ , then (4.51) reduces to the one-breather solution of the modified SP equation presented in Ref. 13. In the limit of small amplitude  $a/b \rightarrow 0$ , (4.51) is approximated by the envelope soliton solution

$$u = \frac{2a}{b^2} \frac{\sin \chi}{\cosh \theta}, \quad v = \frac{\beta}{\alpha} u, \quad (4.52a)$$

$$x = y - \frac{2a}{b^2} \tanh \theta. \quad (4.52b)$$

Another limiting value of  $\delta = \pi/2$  deserves a special attention. To derive a limiting form of the solution, let  $\delta = \pi/2 - \epsilon$ , ( $|\epsilon| \ll 1$ ). It turns out from (4.50) that the permissible values of  $a/b$  lie in the range  $0 < a/b < |\epsilon|/2$ . Then, the parameter  $\chi_0$  from (4.49) is approximated by  $\chi_0 \sim \pi/2 - a/b$ . Inserting these values into (4.48), we obtain the leading order asymptotic of the parametric solution

$$u \sim \frac{2a}{b^2} \sqrt{\frac{\alpha}{\beta}} \frac{(\epsilon - \frac{a}{b}) \sin \chi \cosh \theta - \cos \chi \sinh \theta}{\sinh^2 \theta - \frac{a^2}{b^2} \sin^2 \chi + \epsilon^2}, \quad (4.53a)$$

$$v \sim \frac{2a}{b^2} \sqrt{\frac{\beta}{\alpha}} \frac{(\epsilon + \frac{a}{b}) \sin \chi \cosh \theta + \cos \chi \sinh \theta}{\sinh^2 \theta - \frac{a^2}{b^2} \sin^2 \chi + \epsilon^2}, \quad (4.53b)$$

$$x \sim y - \frac{a}{b^2} \frac{\sinh 2\theta + \frac{a}{b} \sin 2\chi}{\sinh^2 \theta - \frac{a^2}{b^2} \sin^2 \chi + \epsilon^2}. \quad (4.53c)$$

If the parameter  $a/b$  has the same order as  $|\epsilon|$ , then the amplitudes of  $u$  and  $v$  turn out to be of order 1. To be more specific, let  $a/b = |\epsilon|\gamma$  with  $0 < \gamma < 1/2$  and take the limit  $\epsilon \rightarrow 0$ . Then, the solution (4.53) tends to the limiting form

$$u = \frac{2}{b} \sqrt{\frac{\alpha}{\beta}} \frac{(\gamma^{-1} \operatorname{sgn} \epsilon - 1) \sin \hat{\chi} - \hat{\theta} \cos \hat{\chi}}{\hat{\theta}^2 - \sin^2 \hat{\chi} + \gamma^{-2}}, \quad (4.54a)$$

$$v = \frac{2}{b} \sqrt{\frac{\beta}{\alpha}} \frac{(\gamma^{-1} \operatorname{sgn} \epsilon + 1) \sin \hat{\chi} + \hat{\theta} \cos \hat{\chi}}{\hat{\theta}^2 - \sin^2 \hat{\chi} + \gamma^{-2}}, \quad (4.54b)$$

$$x = y - \frac{1}{b} \frac{2\hat{\theta} + \sin 2\hat{\chi}}{\hat{\theta}^2 - \sin^2 \hat{\chi} + \gamma^{-2}}, \quad (4.54c)$$

where  $\hat{\theta} = b(y + \tau/b^2)$ ,  $\hat{\chi} = b(y - \tau/b^2)$  and we have put  $\lambda = \mu = 0$ . In the light of the scale invariance of the system (1.8) mentioned earlier, the above solution is characterized by a single parameter  $\gamma$ .

#### 4. $N$ -soliton solution

The  $N$ -soliton solution consists of a superposition of cusp solitons and breathers. Let  $n$  and  $m$  be the number of cusp solitons and breathers, respectively. Since the soliton parameters appear as complex conjugate pairs for breathers, this type of solutions is realized when the condition  $n + 2m = N$  is satisfied. In particular, for pure breather solutions, we put  $N = 2m$  and impose the following conditions to obtain the  $m$ -breather solution:

$$p_{2j-1} = p_{2j}^* = a_j + ib_j, \quad a_j > 0, \quad b_j > 0, \quad (j = 1, 2, \dots, m), \quad (4.55a)$$

$$\xi_{2j-1} = \theta_j + i\chi_j, \quad \xi_{2j} = \theta_j - i\chi_j, \quad (j = 1, 2, \dots, m), \quad (4.55b)$$

$$a_{2j-1} = \alpha_j e^{i\phi_j} = a_{2j}^*, \quad b_{2j-1} = \beta_j e^{i\psi_j} = b_{2j}^*, \quad (j = 1, 2, \dots, m), \quad (4.55c)$$

where

$$\theta_j = a_j(y + c_j\tau) + \lambda_j, \quad \chi_j = b_j(y - c_j\tau) + \mu_j, \quad c_j = \frac{1}{a_j^2 + b_j^2}, \quad (j = 1, 2, \dots, m). \quad (4.55d)$$

The solution describes multiple collisions of  $m$  smooth breathers provided that the condition similar to (4.50) is imposed on the parameters  $a_j, b_j$  and  $\delta_j (= (\phi_j - \psi_j)/2)$  ( $j = 1, 2, \dots, m$ ). This interesting issue is not discussed here and is left for a future study.

#### D. Remarks

1. If we regard the variables  $u$  and  $v$  as complex-valued functions and impose the condition  $v = u^*$ , then the two-component system (1.8) reduces to a single PDE for the complex variable  $w (\equiv u)$

$$w_{xt} = w + w^*(ww_x)_x, \quad w = w(x, t). \quad (4.56)$$

This equation is a complex version of the modified SP equation. As will be understood, it becomes integrable. Actually, the Lax pair, conservation laws and soliton solutions of Eq. (4.56) can be constructed simply from the corresponding results for Eq. (1.8) by the reduction  $v = u^*$ . In particular, the soliton solution is represented by the parametric representation

$$w = \frac{g}{f}, \quad x = y - (\ln f)_\tau, \quad (4.57)$$

where the tau-functions  $f$  and  $g$  satisfy the system of bilinear equations

$$D_y D_\tau g \cdot f = fg, \quad D_\tau^2 f \cdot f = 2g^* g. \quad (4.58)$$

Note in these expressions that  $f$  should be a real-valued function. Equation (4.57) exhibits breather solutions as well as envelope solitons. For example, the smooth envelope soliton takes the form

$$w = \frac{a}{a^2 + b^2} \frac{e^{ix}}{\cosh \theta}, \quad x = y - \frac{a}{a^2 + b^2} \tanh \theta, \quad (4.59)$$

where  $\theta$  and  $\chi$  are given by (4.49d) and the condition  $0 < a/b < 1$  is imposed to assure the smoothness of the solution. This solution describes the propagation of circularly polarized waves. We remind that the tau-functions for the  $N$ -soliton solution of Eqs. (4.58) have been presented in Ref. 11.

**2.** The relation between Eq. (4.56) and the complex short pulse equation<sup>11,17,18</sup>

$$q_{xt} = q + \frac{1}{2}(|q|^2 q_x)_x, \quad q = q(x, t), \quad (4.60)$$

is worth remarking. One can see that Eq. (4.60) is connected to Eq. (4.56) by the relation (3.18) through the transformations (3.17) with  $n = 2$ , and the identification  $w = u_1 = u_2^*, q = \bar{u}_1 = \bar{u}_2^*$ .

**3.** As a model describing the unidirectional propagation of extremely short pulses in optical fibers, the following equation has been proposed<sup>20–23</sup>

$$E_{xt} = E + (|E|^2 E)_{xx}, \quad E = E(x, t), \quad (4.61)$$

where  $E$  represents a complex variable defined by  $E = E_x + iE_y$  with  $\mathbf{E} = (E_x, E_y)$  being the electric field of the light wave. While the only difference between Eq. (4.60) and Eq. (4.61) is the location of the  $x$  derivative on the right-hand side, the structure of solutions is widely different from each other.

Equation (4.61) admits an envelope soliton of the form

$$E = \frac{\sqrt{6}}{9a} \frac{e^{i\Phi}}{\cosh\left(ay + \frac{t}{9a}\right)}, \quad (4.62a)$$

$$x = y - \frac{2}{3a} \tanh\left(ay + \frac{t}{9a}\right), \quad (4.62b)$$

with

$$\Phi = 2\sqrt{2} \left(ay - \frac{t}{9a}\right) + 2 \tan^{-1} \left[ \sqrt{2} \tanh\left(ay + \frac{t}{9a}\right) \right], \quad (4.62c)$$

where  $a$  is an arbitrary nonzero constant. In accordance with the invariance of Eq. (4.61) under the scale transformation  $x \rightarrow a^{-1}x, t \rightarrow at, E \rightarrow a^{-1}E$ , this constant may be set to 1. One can check by a direct substitution that (4.62) indeed satisfies Eq. (4.61). When compared with an envelope soliton solution of Eq. (4.60) presented in Ref. 11 (see Eq. (4.31)), the solution (4.62) has a complex structure, particularly in the phase variable  $\Phi$  which accounts for the occurrence of the strong phase modulation. One interesting issue to be explored is whether Eq. (4.61) is integrable or not. In this respect, we remark that it has passed the Painlevé test, indicating an evidence of the integrability.<sup>24</sup> Nevertheless, the Lax pair, an infinite number of conservation laws and multisoliton solutions have not been found for it which are common features to integrable systems.

## V. CONCLUDING REMARKS

In this paper, we have shown that the modified SP equation admits an integrable multi-component generalization. In particular, the integrability of the two-component system has been established by constructing its Lax pair. We have also demonstrated that the two-component modified SP equation is transformed to the two-component SP equation proposed by the author through a hodograph transformation. At the level of the bilinear equations, both equations are found to be reduced to the same system of bilinear equations, the only difference being the coordinate transformation. However, this results in a new type of solutions. We have addressed in detail the properties of the cusp solitons and breather solutions for the two-component modified SP equation in which we have derived a condition for the existence of the smooth one-breather solution. We can confirm the existence of smooth multibreather solutions as well numerically by using the analytical solutions. But, its rigorous proof still remains open. Another interesting issue is to generalize Feng's two-component system to the  $n$ -component system with  $n \geq 3$ . Although we have restricted our consideration to the mathematical aspects of the proposed system, the relevance of the system as a model capable of describing the dynamics of ultra-short pulses in optical fibers is an important issue to be studied in a future work.<sup>25</sup>

## ACKNOWLEDGEMENT

This work was supported partially by YAMAGUCHI UNIVERSITY FOUNDATION.

## REFERENCES

1. T. Schäfer and C. E. Wayne, Propagation of ultra-short optical pulses in cubic nonlinear media, *Physica D* **196**, 90-105 (2004).
2. Y. Chung, C.K.R.T. Jones, T. Schäfer and C.E. Wayne, Ultra-short pulses in linear and nonlinear media, *Nonlinearity* **18**, 1351-1374 (2005).
3. M. L. Rabelo, On equations which describe pseudospherical surfaces, *Stud. Appl. Math.* **81**, 221-248 (1989).
4. R. Beals, M. Rabelo and K. Tenenblat, Bäcklund transformation and inverse scattering solutions for some pseudospherical surface equations, *Stud. Appl. Math.* **81**, 125-151 (1989).
5. A. Sakovich and S. Sakovich, The short pulse equation is integrable, *J. Phys. Soc. Jpn.* **74**, 239-241 (2005).
6. J. C. Brunelli, The short pulse hierarchy, *J. Math. Phys.* **46**, 123507 (2005).
7. J.C. Brunelli, The bi-Hamiltonian structure of the short pulse equation, *Phys. Lett. A* **353**, 475-478 (2006).
8. A. Sakovich and S. Sakovich, Solitary wave solutions of the short pulse equation, *J. Phys. A* **39**, L361-L367(2006).
9. Y. Matsuno, Multiloop soliton and multibreather solutions of the short pulse model equation, *J. Phys. Soc. Jpn.* **76**, 084003 (2007).
10. Y. Matsuno, Soliton and periodic solutions of the short pulse model equation, *Handbook of Solitons: Research, Technology and Applications*, edited by S. P. Lang and H. Bedore (Nova, New York, 2009) pp. 541-585.
11. Y. Matsuno, A novel multi-component generalization of the short pulse equation and its multisoliton solutions, *J. Math. Phys.* **52**, 123702 (2011).
12. B.-F. Feng, An Integrable coupled short pulse equation, *J. Phys A: Math. Theor.* **45**, 085202 (2012).
13. S. Sakovich, Transformation and integrability of a generalized short pulse equation, *Commun. Nonlinear Sci. Numer. Simulat.* **39**, 21-28 (2016).
14. Y. Matsuno, *Bilinear Transformation Method* (Academic Press, New York, 1984).
15. R. Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, 2004).
16. R. Hirota and Y. Ohta, Hierarchy of coupled soliton equations. I, *J. Phys. Soc. Jpn.* **60**, 798-809 (1991).

17. A. Dimakis and F. Müller-Hoissen, Bidifferential calculus approach to AKNS hierarchies and their solutions, *SIGMA* **6**, 055 (2010).
18. B.-F. Feng, Complex short pulse and coupled complex short pulse equations, *Physica D* **297**, 62-75 (2015).
19. M. Wadati, H. Sanuki and K. Konno, Relationships among inverse method, Bäcklund transformation and an infinite number of conservation laws, *Prog. Theor. Phys.* **53**, 419-436 (1975).
20. D.V. Kartashov, A.V. Kim and S.A. Skobelev, Soliton structure of a wave field with an arbitrary number of oscillations in nonresonance media, *JETP Lett.* **78**, 276-280 (2003).
21. S.A. Skobelev, D.V. Kartashov and A.V. Kim, Few-optical-cycle solitons and pulse self-compression in a Kerr medium, *Phys. Rev. Lett.* **99**, 203902 (2007).
22. A.V. Kim, S.A. Skobelev, D. Anderson, T. Hansson and M. Lisak, Extreme nonlinear optics in a Kerr medium: Exact soliton solutions for a few cycles, *Phys. Rev. A* **77**, 043823 (2008).
23. Sh. Amiranashvili, A.G. Vladimirov and U. Bandelov, Solitary-wave solutions for few-cycle optical pulses, *Phys. Rev. A* **77**, 063821 (2008).
24. S. Sakovich, Integrability of the vector short pulse equation, *J. Phys. Soc. Jpn.* **77**, 123001 (2008).
25. H. Leblond and D. Mihalache, Models of few optical cycle solitons beyond the slowly varying envelope approximation, *Phys. Rep.* **523**, 61-126 (2013).